

MULTIPLIERS OF FAMILIES OF CAUCHY-STIELTJES TRANSFORMS

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This paper is dedicated to Glenn Schober

ABSTRACT. For $\alpha > 0$ let $\mathcal{F}_\alpha$ denote the class of functions defined for $|z| < 1$ by integrating $1/(1-xz)^\alpha$ against a complex measure on $|x| = 1$. A function g holomorphic in $|z| < 1$ is a multiplier of $\mathcal{F}_\alpha$ if $f \in \mathcal{F}_\alpha$ implies $gf \in \mathcal{F}_\alpha$. The class of all such multipliers is denoted by $\mathcal{M}_\alpha$. Various properties of $\mathcal{M}_\alpha$ are studied in this paper. For example, it is proven that $\alpha < \beta$ implies $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$, and also that $\mathcal{M}_\alpha \subset H^\infty$. Examples are given of bounded functions which are not multipliers. A new proof is given of a theorem of Vinogradov which asserts that if f' is in the Hardy class H^1 , then $f \in \mathcal{M}_1$. Also the theorem is improved to $f' \in H^1$ implies $f \in \mathcal{M}_\alpha$, for all $\alpha > 0$. Finally, let $\alpha > 0$ and let f be holomorphic in $|z| < 1$. It is known that f is bounded if and only if its Cesàro sums are uniformly bounded in $|z| \leq 1$. This result is generalized using suitable polynomials defined for $\alpha > 0$.

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Let $\Delta = \{z: |z| < 1\}$ and $\Gamma = \{z: |z| = 1\}$, and let $\mathcal{M}$ denote the set of complex-valued Borel measures on Γ . For $\alpha > 0$, let $\mathcal{F}_\alpha$ denote the family of functions f for which there exists $\mu \in \mathcal{M}$ such that

$$(1) \quad f(z) = \int_{\Gamma} \frac{1}{(1-xz)^\alpha} d\mu(x), \quad |z| < 1.$$

Here we choose the branch of $1/(1-z)^\alpha$ which equals 1 when $z = 0$.

This class of functions has been studied extensively in the case $\alpha = 1$ [1, 7, 8, 10, 15, 16]. More recently, the families $\mathcal{F}_\alpha$ ($\alpha \neq 1$) were introduced in [13]. Closure properties of the families $\mathcal{F}_\alpha$ were studied by the present authors in [9].

The following two results were proven in [13], and will be useful here.

Theorem A. For $\alpha > 0$, $f \in \mathcal{F}_\alpha$ if and only if $f' \in \mathcal{F}_{\alpha+1}$.

Theorem B. If $f \in \mathcal{F}_\alpha$ and $g \in \mathcal{F}_\beta$, then $fg \in \mathcal{F}_{\alpha+\beta}$.

For $f \in \mathcal{F}_\alpha$, let

$$(2) \quad \|f\|_{\mathcal{F}_\alpha} = \inf\{\|\mu\|: \mu \in \mathcal{M} \text{ such that (1) holds}\}.$$

With this norm, $\mathcal{F}_\alpha$ is a Banach space. As an example, suppose that $f \in \mathcal{F}_\alpha$, μ is a positive measure, and (1) holds. Then $\|f\|_{\mathcal{F}_\alpha} = \|\mu\|$. In the case $\alpha = 1$,

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this was first observed by P. Bourdon and J. A. Cima, who showed in [1] that if $\nu \in \mathcal{M}$ is any other representing measure for f , then

$$\|\mu\| = \mu(\Gamma) = f(0) = \int_{\Gamma} 1 d\nu(x) \leq \|\nu\|.$$

We note that by an easy argument, the infimum in (2) is actually attained.

Let $\{f_n: n = 1, 2, \dots\}$ be a sequence of functions in $\mathcal{F}_\alpha$ and suppose that $f_n \rightarrow f$ in the norm (2). It is easy to show that this implies that $f_n \rightarrow f$ uniformly on compact sets. To see that the converse is false in the case $\alpha = 1$, let $f_n(z) = z^n$ for $|z| < 1$. Then f_n converges uniformly on compact sets to the function $f(z) = 0$. On the other hand, suppose that $\mu_n \in \mathcal{M}$ is any measure representing f_n . Then since

$$z^n = \int_{\Gamma} \frac{1}{1 - xz} d\mu_n(x),$$

it follows that

$$1 = \int_{\Gamma} x^n d\mu_n(x) \leq \int_{\Gamma} 1 d|\mu_n|(x) = \|\mu_n\|.$$

This shows that for each n , $\|f_n\|_{\mathcal{F}_1} \geq 1$, so that the sequence f_n does not converge to f in norm. In the case $\alpha \neq 1$, a similar example can be constructed.

Definition. Suppose that f is holomorphic in Δ . Then f is called a multiplier of $\mathcal{F}_\alpha$ if $g \in \mathcal{F}_\alpha \Rightarrow fg \in \mathcal{F}_\alpha$.

The family of all such multipliers is denoted by $\mathcal{M}_\alpha$.

Suppose that $f \in \mathcal{M}_\alpha$ for some $\alpha > 0$. An application of the Closed Graph Theorem shows that the map $\Lambda: \mathcal{F}_\alpha \rightarrow \mathcal{F}_\alpha$ defined by $\Lambda(g) = fg$ is continuous. Equivalently, Λ is a bounded operator on $\mathcal{F}_\alpha$, so that

$$\sup\{\|fg\|_{\mathcal{F}_\alpha}: g \in \mathcal{F}_\alpha, \|g\|_{\mathcal{F}_\alpha} \leq 1\} < \infty.$$

This last quantity will be denoted by $\|f\|_{\mathcal{M}_\alpha}$, and with this norm $\mathcal{M}_\alpha$ is itself a Banach space.

This paper is concerned with the multiplier families $\mathcal{M}_\alpha$. The family $\mathcal{M}_1$ has been studied in [10], [15], and [16], and various properties of $\mathcal{M}_1$ which were developed there will be generalized to $\mathcal{M}_\alpha$ for $\alpha \neq 1$. For example, S. A. Vinogradov [16] has shown that if f' is in the Hardy space H^1 , then $f \in \mathcal{M}_1$. We give a new proof of this result, and show that if $f' \in H^1$, then $f \in \mathcal{M}_\alpha$, for every $\alpha > 0$. Also we show that if $f \in \mathcal{M}_\alpha$, then f is bounded, and that f has a number of other properties. Examples are given of bounded functions which are not in any $\mathcal{M}_\alpha$ for $\alpha > 0$.

Finally, suppose that f is holomorphic in Δ , and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let

$$\sigma_n(z) = \sum_{j=0}^n \frac{n-j+1}{n+1} a_j z^j.$$

It is a classical result that f is bounded if and only if the Cesàro sums $\sigma_n(z)$ are uniformly bounded for $|z| \leq 1$, and that in this case $\|f\|_{H^\infty} \leq \|\sigma_n\|_{H^\infty}$,

$n = 0, 1, \dots$. This result is generalized here where σ_n is replaced by suitable polynomials depending on $\alpha > 0$.

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In this section various properties of the families $\mathcal{M}_\alpha$ are studied. The following lemma will be useful.

Lemma 2.1. *Let f be holomorphic in Δ , and let $\alpha > 0$. Then $f \in \mathcal{M}_\alpha$ if and only if $f(z)/(1-xz)^\alpha \in \mathcal{F}_\alpha$ for every x with $|x| = 1$ and there exists a constant M such that $\|f(z)/(1-xz)^\alpha\|_{\mathcal{F}_\alpha} \leq M$ for $|x| = 1$.*

Proof. First suppose that $f \in \mathcal{M}_\alpha$. Then multiplication by f is a bounded operator on $\mathcal{F}_\alpha$, and there is a constant M such that

$$(3) \quad \|fg\|_{\mathcal{F}_\alpha} \leq M\|g\|_{\mathcal{F}_\alpha}$$

for all $g \in \mathcal{F}_\alpha$. In particular, (3) holds for all functions of the form $g(z) = 1/(1-xz)^\alpha$, where $|x| = 1$. Since $\|1/(1-xz)^\alpha\|_{\mathcal{F}_\alpha} = 1$, this implies that $\|f(z)/(1-xz)^\alpha\|_{\mathcal{F}_\alpha} \leq M$ for all $|x| = 1$.

For the converse, let $g \in \mathcal{F}_\alpha$. Then for some $\mu \in \mathcal{M}$,

$$g(z) = \int_{\Gamma} \frac{1}{(1-xz)^\alpha} d\mu(x).$$

To show that $fg \in \mathcal{F}_\alpha$, it is enough to consider the case in which μ is a probability measure. Then g is the limit in the topology of uniform convergence on compact subsets of Δ of functions of the form

$$h(z) = \sum_{k=1}^n \mu_k \frac{1}{(1-x_k z)^\alpha}$$

where $\mu_k \geq 0$, $\sum_{k=1}^n \mu_k = 1$, $|x_k| = 1$, and n is a natural number.

For such a function h ,

$$(4) \quad f(z)h(z) = \sum_{k=1}^n \mu_k \frac{f(z)}{(1-x_k z)^\alpha}.$$

By the assumption, there is a measure $\nu_k \in \mathcal{M}$ with $\|\nu_k\| \leq M$ such that

$$\frac{f(z)}{(1-x_k z)^\alpha} = \int_{\Gamma} \frac{1}{(1-xz)^\alpha} d\nu_k(x).$$

Letting $\lambda = \sum_{k=1}^n \mu_k \nu_k$, (4) can be written as

$$f(z)h(z) = \int_{\Gamma} \frac{1}{(1-xz)^\alpha} d\lambda(x),$$

where $\lambda \in \mathcal{M}$ and $\|\lambda\| \leq \sum_{k=1}^n \mu_k \|\nu_k\| \leq M \sum_{k=1}^n \mu_k = M$.

Since $\{\lambda \in \mathcal{M} : \|\lambda\| \leq M\}$ is compact, an argument using subsequences now yields a measure $\sigma \in \mathcal{M}$ with $\|\sigma\| \leq M$ and $f(z)g(z) = \int_{\Gamma} 1/(1-xz)^\alpha d\sigma(x)$. Therefore $fg \in \mathcal{F}_\alpha$, and $f \in \mathcal{M}_\alpha$.

Theorem 2.2. *If $0 < \alpha < \beta$, then $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$.*

Proof. Let $f \in \mathcal{M}_\alpha$. By 2.1, it is enough to show that $f(z)/(1-xz)^\beta \in \mathcal{F}_\beta$ for every x with $|x| = 1$, and to show that there is a constant N such that $\|f(z)/(1-xz)^\beta\|_{\mathcal{F}_\beta} \leq N$, for $|x| = 1$.

Since $f \in \mathcal{M}_\alpha$, the lemma implies that there is a constant M with

$$\|f(z)/(1-xz)^\alpha\|_{\mathcal{F}_\alpha} \leq M, \quad \text{for } |x| = 1.$$

Equivalently, for any x with $|x| = 1$, there is a measure $\mu_x \in \mathcal{M}$ such that

$$(5) \quad \frac{f(z)}{(1-xz)^\alpha} = \int_{\Gamma} \frac{1}{(1-yz)^\alpha} d\mu_x(y)$$

and $\|\mu_x\| \leq M$.

Since

$$\frac{f(z)}{(1-xz)^\beta} = \frac{f(z)}{(1-xz)^\alpha} \frac{1}{(1-xz)^{\beta-\alpha}},$$

(5) yields that

$$\begin{aligned} \frac{f(z)}{(1-xz)^\beta} &= \left\{ \int_{\Gamma} \frac{1}{(1-yz)^\alpha} d\mu_x(y) \right\} \frac{1}{(1-xz)^{\beta-\alpha}} \\ &= \int_{\Gamma} \frac{1}{(1-yz)^\alpha} \frac{1}{(1-xz)^{\beta-\alpha}} du_x(y). \end{aligned}$$

For every x and y with $|x| = |y| = 1$, there is a probability measure $\nu_{x,y}$ such that

$$\frac{1}{(1-yz)^\alpha} \frac{1}{(1-xz)^{\beta-\alpha}} = \int_{\Gamma} \frac{1}{(1-wz)^\beta} d\nu_{x,y}(w) \quad [2, \text{p. 415}].$$

Therefore,

$$\frac{f(z)}{(1-xz)^\beta} = \int_{\Gamma} \int_{\Gamma} \frac{1}{(1-wz)^\beta} d\nu_{x,y}(w) d\mu_x(y).$$

Because $\|\nu_{x,y}\| \leq 1$ and $\|\mu_x\| \leq M$, an argument as in the proof of Lemma 2.1 shows that there is a measure $\lambda \in \mathcal{M}$ with $\|\lambda\| \leq M$ and such that

$$\frac{f(z)}{(1-xz)^\beta} = \int_{\Gamma} \frac{1}{(1-sz)^\beta} d\lambda(s).$$

This shows that $f(z)/(1-xz)^\beta \in \mathcal{F}_\beta$, and that $\|f(z)/(1-xz)^\beta\|_{\mathcal{F}_\beta} \leq M$.

Next we obtain several properties of functions in $\mathcal{M}_\alpha$. First it is shown that such functions are bounded.

Theorem 2.3. *Let $\alpha > 0$ and let $f \in \mathcal{M}_\alpha$. Then $f \in H^\infty$, and $\|f\|_{H^\infty} \leq \|f\|_{\mathcal{M}_\alpha}$.*

Proof. Let M be a constant with $\|f\|_{\mathcal{M}_\alpha} < M$. Let $z_0 = re^{i\theta}$ ($0 \leq r < 1$) and let $x = e^{-i\theta}$.

Since $f \in \mathcal{M}_\alpha$, there is a measure $\mu_x \in \mathcal{M}$ with $\|\mu_x\| < M$ and such that

$$\frac{f(z)}{(1-xz)^\alpha} = \int_{\Gamma} \frac{1}{(1-yz)^\alpha} d\mu_x(y).$$

It follows that

$$(6) \quad f(z) = \int_{\Gamma} \left(\frac{1-xz}{1-yz} \right)^\alpha d\mu_x(y).$$

Letting $z = z_0$ in (6) yields

$$(7) \quad |f(re^{i\theta})| = \left| \int_{\Gamma} \left(\frac{1-r}{1-r\bar{x}y} \right)^{\alpha} d\mu_x(y) \right| \leq \int_{\Gamma} d|\mu_x|(y) < M.$$

Since (7) holds for all r and θ , it follows that $f \in H^{\infty}$ and $\|f\|_{H^{\infty}} < M$, for every M with $M > \|f\|_{\mathcal{M}_{\alpha}}$. Therefore, $\|f\|_{H^{\infty}} \leq \|f\|_{\mathcal{M}_{\alpha}}$.

Theorem 2.4. Let $\alpha > 0$, and let $f \in \mathcal{M}_{\alpha}$. Then $f \in \mathcal{F}_{\alpha}$, and $\|f\|_{\mathcal{F}_{\alpha}} \leq \|f\|_{\mathcal{M}_{\alpha}}$.

Proof. Let $I(z) = 1$ for $|z| < 1$. Since

$$I(z) = \int_{\Gamma} \frac{1}{(1-xz)^{\alpha}} dm(x),$$

where m denotes normalized Lebesgue measure, $I \in \mathcal{F}_{\alpha}$. Also, since m is a positive measure, the remark in §1 shows that

$$(8) \quad \|I\|_{\mathcal{F}_{\alpha}} = \|m\| = 1.$$

Since $f \in \mathcal{M}_{\alpha}$ and $I \in \mathcal{F}_{\alpha}$, it follows that $f = fI \in \mathcal{F}_{\alpha}$. Also, since

$$\|f\|_{\mathcal{F}_{\alpha}} = \|fI\|_{\mathcal{F}_{\alpha}} \leq \|f\|_{\mathcal{M}_{\alpha}} \|I\|_{\mathcal{F}_{\alpha}},$$

(8) implies that

$$(9) \quad \|f\|_{\mathcal{F}_{\alpha}} \leq \|f\|_{\mathcal{M}_{\alpha}}. \quad \square$$

We note that the inequality (9) is sharp, because $I \in \mathcal{M}_{\alpha}$ and $\|I\|_{\mathcal{F}_{\alpha}} = 1$.

As an application of Theorem 2.4, let

$$(10) \quad \frac{1}{(1-z)^{\alpha}} = \sum_{n=0}^{\infty} A_n(\alpha) z^n \quad (|z| < 1),$$

and suppose that $f \in \mathcal{M}_{\alpha}$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$). The theorem asserts that for some $\mu \in \mathcal{M}$,

$$(11) \quad f(z) = \int_{\Gamma} \frac{1}{(1-xz)^{\alpha}} d\mu(x).$$

Equations (10) and (11) imply that

$$a_n = A_n(\alpha) \int_{\Gamma} x^n d\mu(x).$$

Since $A_n(\alpha) = O(n^{\alpha-1})$, and since $|\int_{\Gamma} x^n d\mu(x)| \leq \|\mu\|$, this shows that the coefficients of f obey $|a_n| = O(n^{\alpha-1})$.

In the case $0 < \alpha < 1$, this coefficient estimate provides additional information on functions in $\mathcal{M}_{\alpha}$. Suppose that f is holomorphic in Δ , and that $f(z) = \sum_{n=0}^{\infty} a_n z^n$. In [16] it was shown that if $\sum_{n=0}^{\infty} |a_n| \log(n+2) < \infty$, then $f \in \mathcal{M}_1$. In particular, the function $f(z) = \sum_{n=0}^{\infty} (1/n^3) z^{2^n}$ is in $\mathcal{M}_1$, but for $m = 2^n$, $a_m \neq O(m^{\alpha-1})$, for each α ($0 < \alpha < 1$). This shows that $f \notin \mathcal{M}_{\alpha}$ for $\alpha < 1$. The first author and E. A. Nordgren have shown that $\mathcal{M}_1 \neq \mathcal{M}_2$, and also that for $0 < \alpha < \beta < 1$, $\mathcal{M}_{\alpha} \neq \mathcal{M}_{\beta}$. It is an open question to determine if $\mathcal{M}_{\alpha} \neq \mathcal{M}_{\beta}$ for all $\alpha \neq \beta$.

It was shown in [9] that $\mathcal{F}_{\alpha}$ is closed under composition with disk automorphisms $z \rightarrow (z + \xi)/(1 + \bar{\xi}z)$, where $|\xi| < 1$. This will be used in the proof of the next theorem, which asserts the same result for $\mathcal{M}_{\alpha}$.

Theorem 2.5. Let $\alpha > 0$. If $f \in \mathcal{M}_\alpha$, $|\xi| < 1$, and $g(z) = f((z + \xi)/(1 + \bar{\xi}z))$, then $g \in \mathcal{M}_\alpha$.

Proof. Let $h \in \mathcal{F}_\alpha$, and let $k(z) = h((z - \xi)/(1 - \bar{\xi}z))$. Since the map $w = (z - \xi)/(1 - \bar{\xi}z)$ is an automorphism of Δ , the result in [9] quoted above shows that $k \in \mathcal{F}_\alpha$. Since $f \in \mathcal{M}_\alpha$, it follows that $m = fk \in \mathcal{F}_\alpha$. A second application of the result in [9] implies that $m((z + \xi)/(1 + \bar{\xi}z)) \in \mathcal{F}_\alpha$. Since

$$m\left(\frac{z + \xi}{1 + \bar{\xi}z}\right) = f\left(\frac{z + \xi}{1 + \bar{\xi}z}\right)k\left(\frac{z + \xi}{1 + \bar{\xi}z}\right) = g(z)h(z),$$

this shows that $g \in \mathcal{M}_\alpha$.

The following theorem generalizes a result in [16], which showed that if $f \in \mathcal{M}_1$, then f has finite radial variation in every direction.

Theorem 2.6. For each $\alpha > 0$ there is a constant A_α such that if $f \in \mathcal{M}_\alpha$, then the radial variation of f in the direction θ obeys $V(f, \theta) \leq A_\alpha \|f\|_{\mathcal{M}_\alpha}$ for all θ .

Proof. Suppose that $f \in \mathcal{M}_\alpha$ for some $\alpha > 0$. If $|\xi| = 1$ then there is a measure μ_ξ such that

$$(12) \quad f(z) \frac{1}{(1 - \xi z)^\alpha} = \int_{\Gamma} \frac{1}{(1 - xz)^\alpha} d\mu_\xi(x).$$

Also, if $M = \|f\|_{\mathcal{M}_\alpha}$, and $\varepsilon > 0$, then $\|\mu_\xi\| \leq M + \varepsilon$ for $|\xi| = 1$.

It follows from (12) that

$$f'(z) = \alpha \int_{\Gamma} \frac{(1 - \xi z)^{\alpha-1} (x - \xi)}{(1 - xz)^{\alpha+1}} d\mu_\xi(x),$$

and therefore

$$(13) \quad \int_0^1 |f'(r\bar{\xi})| dr \leq \alpha \int_{\Gamma} \left[\int_0^1 \frac{(1-r)^{\alpha-1} |x - \xi|}{|1 - r x \bar{\xi}|^{\alpha+1}} dr \right] d|\mu_\xi|(x).$$

Let I denote the inner integral on the right-hand side of (13). Because

$$\begin{aligned} |1 - r x \bar{\xi}|^{\alpha+1} &= \{ |1 - r x \bar{\xi}|^2 \}^{(\alpha+1)/2} = \{ (1-r)^2 + r |1 - x \bar{\xi}|^2 \}^{(\alpha+1)/2} \\ &\geq \{ (1-r)^2 + r^2 |1 - x \bar{\xi}|^2 \}^{(\alpha+1)/2}, \end{aligned}$$

it follows that

$$I \leq \int_0^1 \frac{(1-r)^{\alpha-1} b}{\{ (1-r)^2 + r^2 b^2 \}^{(\alpha+1)/2}} dr \equiv J,$$

where $b = |1 - x \bar{\xi}|$. The change of variables $y = rb/(1-r)$ shows that $J = \int_0^\infty \frac{1}{(1+y^2)^{(\alpha+1)/2}} dy \equiv B_\alpha$. This integral converges since $\int_1^\infty 1/y^\beta dy$ converges for $\beta > 1$. Therefore (13) yields that

$$\int_0^1 |f'(r\bar{\xi})| dr \leq \alpha \int_{\Gamma} B_\alpha d|\mu_\xi|(x) \leq A_\alpha (M + \varepsilon),$$

where $A_\alpha = \alpha B_\alpha$. Let $\varepsilon \rightarrow 0$, the theorem is established. $\square$

Let $f \in \mathcal{M}_\alpha$. As a consequence of Theorem 2.6, the radial limit $\lim_{r \rightarrow 1} f(re^{i\theta})$ exists for all θ . Also, note that the conclusion of the theorem implies that f is bounded.

As an application of Theorem 2.6, we next give a number of simple examples of bounded functions which are not in $\mathcal{M}_\alpha$ for any $\alpha > 0$.

As a first example, let $f(z) = (1 - z)^{-i}$, using the principal branch of the logarithm. Then f is holomorphic in Δ , and since $|f(z)| = e^{-\text{Arg}(1-z)}$, it follows that $|f(z)| < e^{\pi/2}$ for $|z| < 1$. It is easy to verify that f maps the interval $[0, 1)$ onto the circle Γ covered infinitely often and hence the curve $w = f(r)$, $0 \leq r < 1$, is not rectifiable. It follows by Theorem 2.6 that $f \notin \mathcal{M}_\alpha$ for any $\alpha > 0$.

In [9], it was shown that if f is holomorphic in $\bar{\Delta}$, then $f \in \mathcal{M}_\alpha$ for all $\alpha > 0$. In particular, this implies that a finite Blaschke product belongs to $\mathcal{M}_\alpha$ for $\alpha > 0$. Theorem 2.5 provides a second proof of this fact, as follows. Let $I(z) = z$ for $|z| < 1$. It is clear that $I \in \mathcal{M}_\alpha$ for $\alpha > 0$. If $|\xi| < 1$, then Theorem 2.5 implies that

$$I\left(\frac{z + \xi}{1 + \bar{\xi}z}\right) = \frac{z + \xi}{1 + \bar{\xi}z} \in \mathcal{M}_\alpha, \quad \text{for } \alpha > 0.$$

Since the finite product of functions in $\mathcal{M}_\alpha$ is itself in $\mathcal{M}_\alpha$, this proves the assertion.

We next show that there are infinite Blaschke products which are not in $\mathcal{M}_\alpha$ for any $\alpha > 0$. Let $f(z) = \prod_{n=1}^{\infty} (a_n - z)/(1 - \bar{a}_n z)$ where $a_n = 1 - 1/2^n$, $n = 1, 2, \dots$. In [6] it was shown that there is a constant $A > 0$ such that if $\rho_n = \frac{1}{2}(a_n + a_{n+1})$ then $|f(\rho_n)| \geq A$ for $n = 1, 2, \dots$. It follows that $\int_0^1 |f'(r)| dr = \infty$, so that by Theorem 2.6, $f \notin \mathcal{M}_\alpha$ for $\alpha > 0$.

We note that in [10], it was proved that an inner function belongs to $\mathcal{M}_1$ if and only if it is a Blaschke product with the sequence of zeros satisfying the Frostman condition.

The next example shows that a function holomorphic in Δ and continuous in $\bar{\Delta}$ need not be in $\mathcal{M}_\alpha$ for any $\alpha > 0$. In [17], L. Zalcman described a bounded region D such that ∂D is a Jordan curve, $z = 1 \in \partial D$, and $z = 1$ is not rectifiably accessible from the interior of D . Since ∂D is a Jordan curve, any conformal mapping of Δ onto D extends continuously to $\bar{\Delta}$. Let f be such a map with $f(1) = 1$. Then $f \notin \mathcal{M}_\alpha$, since the curve $w = f(r)$, $0 \leq r \leq 1$, is not rectifiable. The argument in [17] even shows that the power series for f is uniformly convergent on $\partial\Delta$. Hence even with this additional condition we can still have $f \notin \mathcal{M}_\alpha$ for all $\alpha > 0$.

The examples above give bounded functions for which the radial variation in one direction is infinite. A stronger result is presented in [14], where examples are given of infinite Blaschke products $B(z)$ for which the radial variation $V(B, \theta) = \infty$ for almost all θ . Also, [14] includes the construction of a function f holomorphic in Δ and continuous in $\bar{\Delta}$ for which $V(f, \theta) = \infty$ for almost all θ .

3

In this section a condition is shown to be sufficient for membership in $\mathcal{M}_\alpha$ for every $\alpha > 0$. Let H^1 denote the Hardy space of functions f that are holomorphic in Δ and such that

$$\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})| d\theta < \infty.$$

In [16, p. 20] it was proved by Vinogradov that if $f' \in H^1$ then $f \in \mathcal{M}_1$. This result is generalized to $f' \in H^1$ implies $f \in \mathcal{M}_\alpha$ for every $\alpha > 0$. This strengthens the result in [9] which asserts that if f is holomorphic in $\bar{\Delta}$ then $f \in \mathcal{M}_\alpha$ for every $\alpha > 0$.

We begin by giving a new proof of Vinogradov's theorem. It may have independent interest especially since it shows that this result is related to the class of functions of bounded mean oscillation [5, p. 222]. Let $\mathcal{B}$ denote the set of functions f holomorphic in Δ which can be expressed as $f = g + h$, where g and h are holomorphic in Δ , $\operatorname{Re} g$ is bounded in Δ , and $\operatorname{Im} h$ is bounded in Δ . If $f \in \mathcal{B}$ then $\|f\|_{\mathcal{B}}$ is defined by $\inf(\|\operatorname{Re} g\|_\infty + \|\operatorname{Im} h\|_\infty)$ where g and h vary over all pairs as above. Here $\|u\|_\infty = \sup_{|z|<1} |u(z)|$ for any function u defined in Δ .

Lemma 3.1. *Let f be holomorphic in Δ and suppose that there is a holomorphic function g and a constant $M > 0$ such that*

$$(14) \quad |f(z) + g(\bar{z})| \leq M \quad \text{for } |z| < 1.$$

Then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \leq M$.

Proof. Let $s = \operatorname{Re} f$, $t = \operatorname{Im} f$, $u = \operatorname{Re} g$, and $v = \operatorname{Im} g$. The function G defined by $G(z) = \frac{1}{2}[f(z) + g(\bar{z})]$ is holomorphic in Δ and $\operatorname{Re} G(z) = \frac{1}{2}[s(z) + u(\bar{z})]$. Hence (14) implies that $|\operatorname{Re} G(z)| \leq \frac{1}{2}M$ for $|z| < 1$. The function H defined by $H(z) = \frac{1}{2}[f(z) - \overline{g(\bar{z})}]$ is holomorphic in Δ and $\operatorname{Im} H(z) = \frac{1}{2}[t(z) + v(\bar{z})]$. Hence (14) implies $|\operatorname{Im} H(z)| \leq \frac{1}{2}M$ for $|z| < 1$. Since $f = G + H$ this yields $f \in \mathcal{B}$. Moreover $\|f\|_{\mathcal{B}} \leq \|\operatorname{Re} G\|_\infty + \|\operatorname{Im} H\|_\infty \leq M$.

Lemma 3.2. *Let $f \in H^\infty$ and let g be defined by*

$$(15) \quad g(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1-w} dw$$

for $|z| < 1$. Then $|g'(z)| \leq B\|f\|_{H^\infty}/|1-z|$ for $|z| < 1$, where B is an absolute constant.

Proof. We first show that if $|z| < 1$ and α is the line segment from $w = 0$ to $w = z$ then

$$(16) \quad \int_\alpha \frac{1}{|1-w|^2} |dw| \leq \frac{\pi}{2} \frac{|z|}{|1-z|}.$$

This is clear if $z = 0$. Also if z is real and $z \neq 0$ then we have

$$\int_\alpha \frac{1}{|1-w|^2} |dw| = |z| \int_0^1 \frac{1}{(1-tz)^2} dt = \frac{|z|}{1-z},$$

and hence (16) follows. Henceforth assume that $|z| < 1$ and z is not real. Then

$$\begin{aligned} \int_\alpha \frac{1}{|1-w|^2} |dw| &= |z| \int_0^1 \frac{1}{(1-tz)(1-t\bar{z})} dt \\ &= \frac{|z|}{z-\bar{z}} \left\{ \log \frac{1}{1-z} - \log \frac{1}{1-\bar{z}} \right\} \\ &= \frac{|z|}{z-\bar{z}} \int_\beta \frac{1}{1-w} dw \end{aligned}$$

where β is the arc on the circle that is centered at $w = 1$ and goes from $\bar{z}$ to z . Let θ denote the angle subtended by the arc β and let L denote the length of β . Then $|z - \bar{z}| = 2|1 - z|\sin(\theta/2)$ and $L = |1 - z|\theta$. Therefore

$$\int_{\alpha} \frac{1}{|1 - w|^2} |dw| \leq \frac{|z|}{|z - \bar{z}|} \frac{1}{|1 - z|} L = \frac{\theta/2}{\sin(\theta/2)} \frac{|z|}{|1 - z|} \leq \frac{\pi}{2} \frac{|z|}{|1 - z|},$$

since $0 < \theta/2 \leq \pi/2$. This proves (16).

From (15) we obtain $zg'(z) + g(z) = f(z)/(1 - z)$. Hence an integration by parts yields

$$\begin{aligned} z^2 g'(z) &= \frac{zf(z)}{1 - z} - zg(z) = \frac{zf(z)}{1 - z} - \int_0^z \frac{f(w)}{1 - w} dw \\ &= \frac{zf(z)}{1 - z} - \frac{h(z)}{1 - z} + \int_0^z \frac{h(w)}{(1 - w)^2} dw, \end{aligned}$$

where

$$(17) \quad h(z) = \int_0^z f(w) dw$$

for $|z| < 1$. Clearly (17) implies $\|h\|_{H^\infty} \leq \|f\|_{H^\infty} \equiv M$. It follows that

$$|z^2 g'(z)| \leq \frac{M}{|1 - z|} + \frac{M}{|1 - z|} + M \int_{\alpha} \frac{1}{|1 - w|^2} |dw|.$$

Therefore (16) implies that

$$|z^2 g'(z)| \leq \left(2 + \frac{\pi}{2}\right) M \frac{1}{|1 - z|} \quad \text{for } |z| < 1.$$

The function G defined by $G(z) = (1 - z)z^2 g'(z)$ is analytic in Δ , has at least a second order zero at $z = 0$ and satisfies $|G(z)| \leq BM$ for $|z| < 1$ where $B = 2 + \pi/2$. Hence $|G(z)| \leq BM|z|^2$ for $|z| < 1$ and therefore $|g'(z)| \leq BM/|1 - z|$. $\square$

Lemma 3.3. Suppose that $f \in H^\infty$ and g is defined by

$$(18) \quad g(z) = \frac{1}{z} \int_0^z \frac{f(w)}{1 - w} dw$$

for $|z| < 1$. Then $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \leq A\|f\|_{H^\infty}$ where A is an absolute constant.

Proof. By equation (18) and Lemma 3.2, there is an absolute constant B such that

$$(19) \quad |g'(z)| \leq \frac{B\|f\|_{H^\infty}}{|1 - z|} \quad \text{for } |z| < 1.$$

Let $|z| < 1$ and let γ denote the circle centered at 1 which passes through z and has radius $r = |1 - z|$. Let δ denote the subarc of γ from $\bar{z}$ to z . Then

$$g(z) - g(\bar{z}) = \int_{\delta} g'(w) dw$$

and hence (19) implies that

$$|g(z) - g(\bar{z})| \leq \frac{B\|f\|_{H^\infty}}{r} (\text{length of } \delta) \leq \frac{\pi}{2} B\|f\|_{H^\infty}.$$

An application of Lemma 3.1 in the special case where the functions there are related by $g = -f$ implies that $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \leq A\|f\|_{H^\infty}$ where $A = \pi B/2$. $\square$

Lemma 3.4. Suppose that f and g are functions holomorphic in $\bar{\Delta}$ and let F and G be defined by

$$(20) \quad F(z) = \frac{1}{1-z} \int_z^1 f(w) dw$$

and

$$(21) \quad G(z) = \frac{1}{z} \int_0^z \frac{1}{1-w} g(w) dw.$$

Then

$$(22) \quad \int_0^{2\pi} f(e^{i\theta}) G(e^{-i\theta}) d\theta = \int_0^{2\pi} F(e^{i\theta}) g(e^{-i\theta}) d\theta.$$

Proof. There is a number $R > 1$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ for $|z| < R$. Then F also is holomorphic in $\{z: |z| < R\}$ and G is holomorphic in $\bar{\Delta}$ except possibly for a logarithmic singularity at $z = 1$. In particular, $G \in H^1$ (in fact, $G \in H^p$ for all $p > 0$). For $|z| < R$ we have

$$\begin{aligned} F(z) &= \frac{1}{1-z} \int_z^1 \left(\sum_{n=0}^{\infty} a_n w^n \right) dw = \frac{1}{1-z} \sum_{n=0}^{\infty} \frac{a_n}{n+1} (1 - z^{n+1}) \\ &= \sum_{n=0}^{\infty} \left\{ \frac{a_n}{n+1} \sum_{k=0}^n z^k \right\} = \sum_{n=0}^{\infty} \left\{ \sum_{k=n}^{\infty} \frac{a_k}{k+1} \right\} z^n. \end{aligned}$$

Therefore

$$(23) \quad \int_0^{2\pi} F(e^{i\theta}) g(e^{-i\theta}) d\theta = 2\pi \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} \frac{a_k}{k+1} \right) b_n = 2\pi \sum_{n=0}^{\infty} \left\{ \frac{a_n}{n+1} \sum_{k=0}^n b_k \right\}.$$

For $|z| < 1$, we have

$$\begin{aligned} G(z) &= \frac{1}{z} \int_0^z \left(\sum_{n=0}^{\infty} w^n \right) \left(\sum_{n=0}^{\infty} b_n w^n \right) dw \\ &= \frac{1}{z} \int_0^z \left(\sum_{n=0}^{\infty} \left(\sum_{k=0}^n b_k \right) w^n \right) dw \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^n b_k \right) z^n. \end{aligned}$$

If $0 < r < 1$ then

$$\int_0^{2\pi} f(e^{i\theta}) G(re^{-i\theta}) d\theta = 2\pi \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} \sum_{k=0}^n b_k \right) r^n \equiv H(r).$$

Since the series defining H converges at $r = 1$, Abel's theorem gives

$$(24) \quad \lim_{r \rightarrow 1-} \int_0^{2\pi} f(e^{i\theta}) G(re^{-i\theta}) d\theta = \lim_{r \rightarrow 1-} H(r) = 2\pi \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} \sum_{k=0}^n b_k \right).$$

Also, because $f(e^{i\theta})$ is bounded and $G \in H^1$ it follows that

$$(25) \quad \lim_{r \rightarrow 1-} \int_0^{2\pi} f(e^{i\theta}) G(re^{-i\theta}) d\theta = \int_0^{2\pi} f(e^{i\theta}) G(e^{-i\theta}) d\theta.$$

Therefore by (23), (24), and (25),

$$\int_0^{2\pi} F(e^{i\theta}) g(e^{-i\theta}) d\theta = 2\pi \sum_{n=0}^{\infty} \left(\frac{a_n}{n+1} \sum_{k=0}^n b_k \right) = \int_0^{2\pi} f(e^{i\theta}) G(e^{-i\theta}) d\theta. \quad \square$$

We thank D. J. Hallenbeck for pointing out and rectifying an error in our initial proof of Lemma 3.4.

Theorem C (Vinogradov). *If $f' \in H^1$, then $f \in \mathcal{M}_1$.*

Proof. Suppose that $f' \in H^1$ and $|\xi| = 1$. We first note that

$$\frac{f(z)}{\xi - z} = \frac{1}{\xi} \frac{f(z)}{1 - \bar{\xi}z}.$$

Therefore by Lemma 2.1, it is enough to show that $f(z)/(\xi - z) \in \mathcal{F}_1$, and that there is a constant $M > 0$ such that $\|f(z)/(\xi - z)\|_{\mathcal{F}_1} \leq M$ for all $|\xi| = 1$. Also note that

$$\frac{f(z)}{\xi - z} = \frac{1}{\xi - z} \int_0^z f'(w) dw + \frac{f(0)}{\xi - z}.$$

Since $f(0)/(\xi - z) \in \mathcal{F}_1$ and since $\|f(0)/(\xi - z)\|_{\mathcal{F}_1} = |f(0)|$, it suffices to show that the function $(\xi - z)^{-1} \int_0^z f'(w) dw$ belongs to $\mathcal{F}_1$ and that for some $M > 0$, $\|(\xi - z)^{-1} \int_0^z f'(w) dw\|_{\mathcal{F}_1} \leq M$ for all $|\xi| = 1$. The argument is carried out with $\xi = 1$ and a similar argument serves for all ξ providing the same bound on the norm.

In our formulation we replace f' by f . In other words, assume that $f \in H^1$ and let

$$(26) \quad g(z) = \frac{1}{1-z} \int_0^z f(w) dw \quad \text{for } |z| < 1.$$

Then $g(z) = b/(1-z) - (1-z)^{-1} \int_z^1 f(w) dw$, where $b = \int_0^1 f(w) dw$.

First note that

$$\begin{aligned} |b| &\leq \int_0^1 |f(w)| |dw| \leq \int_{-1}^1 |f(w)| |dw| \\ &\leq \frac{1}{2} \int_0^{2\pi} |f(e^{i\theta})| d\theta = \pi \|f\|_{H^1} \quad [4, \text{p. 46}]. \end{aligned}$$

It follows that

$$(27) \quad \left\| \frac{b}{1-z} \right\|_{\mathcal{F}_1} \leq \pi \|f\|_{H^1}$$

Next let $k(z) = (1-z)^{-1} \int_z^1 f(w) dw$. Let $\mathcal{A}$ denote the space of functions holomorphic in Δ and continuous in $\bar{\Delta}$. To show that $k \in \mathcal{F}_1$ it suffices to prove that there is a constant $A > 0$ such that

$$(28) \quad \left| \int_0^{2\pi} k(re^{i\theta}) h(e^{-i\theta}) d\theta \right| \leq A \|h\|_{H^\infty}$$

for $0 < r < 1$ and for all $h \in A$. This inequality will be obtained where $A = B\|f\|_{H^1}$ and B is an absolute constant. This will imply that

$$(29) \quad \|k\|_{\mathcal{H}_1} \leq B\|f\|_{H^1}$$

and it then follows from (26), (27), and (29) that $\|g\|_{\mathcal{H}_1} \leq (\pi + B)\|f\|_{H^1}$.

By first making the change of variables $z \rightarrow \rho z$ where $0 < \rho < 1$ and then letting $\rho \rightarrow 1$, we may assume that f and h are holomorphic in $\bar{\Delta}$. Then k is holomorphic in $\bar{\Delta}$. We now show that it suffices to prove that

$$(30) \quad \left| \int_0^{2\pi} k(e^{i\theta})h(e^{-i\theta})d\theta \right| \leq C\|f\|_{H^1}\|h\|_{H^\infty},$$

where C is an absolute constant. For $0 \leq r \leq 1$ let $F(r) = \int_0^{2\pi} k(re^{i\theta})h(e^{i\theta})d\theta$. Assuming (30) we get $|F(1)| \leq C\|f\|_{H^1}\|h\|_{H^\infty}$. Since F is continuous in $[0, 1]$, there exists r_0 ($0 < r_0 < 1$) such that $|F(r)| \leq 2|F(1)|$ for $r_0 \leq r \leq 1$. Therefore

$$(31) \quad |F(r)| \leq 2C\|f\|_{H^1}\|h\|_{H^\infty} \quad \text{for } r_0 \leq r < 1.$$

Suppose now that $0 \leq r \leq r_0$. Then

$$|F(r)| \leq \int_0^{2\pi} |k(re^{i\theta})||h(e^{-i\theta})|d\theta \leq \|h\|_{H^\infty} \int_0^{2\pi} |k(r_0e^{i\theta})|d\theta.$$

Without loss of generality we may assume that $f \neq 0$. Then $k \neq 0$, $\|f\|_{H^1} > 0$, and $\int_0^{2\pi} |k(r_0e^{i\theta})|d\theta > 0$. Therefore for some $D > 0$, $\int_0^{2\pi} |k(r_0e^{i\theta})|d\theta = D\|f\|_{H^1}$. It follows that

$$(32) \quad |F(r)| \leq D\|f\|_{H^1}\|h\|_{H^\infty} \quad \text{for } 0 \leq r \leq r_0.$$

Letting $B = \max(2C, D)$, relations (31) and (32) imply that

$$|F(r)| \leq B\|f\|_{H^1}\|h\|_{H^\infty} \quad \text{for } 0 \leq r \leq 1.$$

This proves (28).

It remains to prove the assertion (30). Let $m(z) = z^{-1} \int_0^z (1-w)^{-1}h(w)dw$. Lemma 3.3 implies that $m \in \mathcal{B}$ and $\|m\|_{\mathcal{B}} \leq C\|h\|_{H^\infty}$ for an absolute constant C . We have $m = p + q$ where p and q are holomorphic in Δ and $u = \operatorname{Re} p$ and $v = \operatorname{Im} q$ are bounded and $\|u\|_\infty + \|v\|_\infty \leq C\|h\|_{H^\infty}$. Now

$$\int_0^{2\pi} f(e^{i\theta})m(e^{-i\theta})d\theta = \int_0^{2\pi} f(e^{i\theta})p(e^{-i\theta})d\theta + \int_0^{2\pi} f(e^{i\theta})q(e^{-i\theta})d\theta.$$

Using power series and the orthonormal relations for the trigonometric functions, this equals

$$\int_0^{2\pi} f(e^{i\theta})u(e^{-i\theta})d\theta + i \int_0^{2\pi} f(e^{i\theta})v(e^{-i\theta})d\theta.$$

Hence

$$\begin{aligned} \left| \int_0^{2\pi} f(e^{i\theta})m(e^{-i\theta})d\theta \right| &\leq \|u\|_\infty\|f\|_{H^1} + \|v\|_\infty\|f\|_{H^1} \\ &= (\|u\|_\infty + \|v\|_\infty)\|f\|_{H^1} \\ &\leq C\|f\|_{H^1}\|h\|_{H^\infty}. \end{aligned}$$

Because of Lemma 3.4, this yields

$$\left| \int_0^{2\pi} k(e^{i\theta})h(e^{-i\theta})d\theta \right| \leq C\|f\|_{H^1}\|h\|_{H^\infty},$$

which is the required inequality. $\square$

The argument used to prove Theorem C does not depend on the duality theorem about H^1 and BMO proved by C. Fefferman [5, p. 245]. It is interesting to note that the function g defined in Lemma 3.3 can be shown to have bounded mean oscillation by a fairly direct argument.

The essential ideas for proving Theorem C as developed above are due to Boris Korenblum [12]. The authors would like to thank Korenblum for several helpful conversations about multipliers.

Theorem 3.5. *If $f' \in H^1$, then $f \in \mathcal{M}_\alpha$ for all $\alpha > 0$.*

Proof. Let $f' \in H^1$. By Theorem C, $f \in \mathcal{M}_1$, and by Theorem 2.2 it follows that $f \in \mathcal{M}_\alpha$ for every $\alpha > 1$.

In the case $0 < \alpha < 1$, let $g \in \mathcal{F}_\alpha$, and let $h = fg$. By Theorem A, it suffices to show that $h' \in \mathcal{F}_{\alpha+1}$.

Since $g \in \mathcal{F}_\alpha$, Theorem A implies that $g' \in \mathcal{F}_{\alpha+1}$. By the previous part of the proof, $f \in \mathcal{M}_{\alpha+1}$, and therefore

$$(33) \quad fg' \in \mathcal{F}_{\alpha+1}.$$

Because $f' \in H^1$, it follows that $f' \in \mathcal{F}_1$ [4, p. 34]. By assumption, $g \in \mathcal{F}_\alpha$ and so Theorem B implies that

$$(34) \quad f'g \in \mathcal{F}_{\alpha+1}.$$

Since $h' = fg' + f'g$, (33) and (34) show that $h' \in \mathcal{F}_{\alpha+1}$, or equivalently, $h \in \mathcal{F}_\alpha$. This proves that $f \in \mathcal{M}_\alpha$ for $0 < \alpha < 1$. $\square$

Theorem 3.5 is sharp, since there are functions f such that $f' \in H^p$ ($0 < p < 1$) and f is not bounded. By Theorem 2.3, such functions are not multipliers.

One example where Theorem 3.5 applies concerns bounded convex maps. Suppose that f is holomorphic in Δ and that f maps Δ one-to-one onto a bounded convex region. Since the boundary C of such a region is rectifiable and since C is a Jordan curve, it follows that $f' \in H^1$ [4, p. 44]. Therefore, $f \in \mathcal{M}_\alpha$, for $\alpha > 0$.

4

Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic in Δ . Let

$$s_n(z) = \sum_{j=0}^n a_j z^j$$

and

$$\sigma_n(z) = \frac{1}{n+1} \sum_{j=0}^n s_j(z).$$

By a classical result [3, p. 439], the function f is bounded if and only if the sequence $\sigma_n(z)$ is uniformly bounded for $n = 0, 1, \dots$ and for $|z| \leq 1$, and

in this case, $\|f\|_{H^\infty} = \sup\{\|\sigma_n\|_{H^\infty} : n = 0, 1, \dots\}$. This result is generalized in this section, in terms of polynomials which are generated in the study of the multiplier problem.

Definition. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ ($|z| < 1$), let

$$P_n(z; \alpha) = \frac{1}{A_n(\alpha)} \{A_n(\alpha)a_0 + A_{n-1}(\alpha)a_1 z + \dots + A_1(\alpha)a_{n-1} z^{n-1} + A_0(\alpha)a_n z^n\}$$

where $\alpha > 0$, $n = 0, 1, \dots$, and $z \in \mathbb{C}$.

Theorem 4.1. If $f \in \mathcal{M}_\alpha$, then $\|P_n(z; \alpha)\|_{H^\infty} \leq \|f\|_{\mathcal{M}_\alpha}$ for $n = 0, 1, \dots$.

Proof. Let $f \in \mathcal{M}_\alpha$ and suppose that $M > \|f\|_{\mathcal{M}_\alpha}$. If $|x| = 1$ then we have $f(z)/(1-xz)^\alpha \in \mathcal{F}_\alpha$. Also,

$$\left\| f(z) \frac{1}{(1-xz)^\alpha} \right\|_{\mathcal{F}_\alpha} \leq M \quad \text{for all } |x| = 1.$$

Therefore for each x ($|x| = 1$) there is a measure $\mu_x \in \mathcal{M}$ such that

$$(35) \quad f(z) \frac{1}{(1-xz)^\alpha} = \int_{\Gamma} \frac{1}{(1-yz)^\alpha} d\mu_x(y),$$

and $\|\mu_x\| \leq M$ for $|x| = 1$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f(z)/(1-xz)^\alpha = \sum_{n=0}^{\infty} b_n z^n$ where

$$b_n = A_0(\alpha)a_n + A_1(\alpha)a_{n-1}x + \dots + A_{n-1}(\alpha)a_1 x^{n-1} + A_n(\alpha)a_0 x^n.$$

If

$$\int_{\Gamma} \frac{1}{(1-yz)^\alpha} d\mu_x(y) = \sum_{n=0}^{\infty} c_n z^n,$$

then

$$c_n = A_n(\alpha) \int_{\Gamma} y^n d\mu_x(y).$$

Because of (35), $b_n = c_n$, or

$$(36) \quad x^n P_n\left(\frac{1}{x}; \alpha\right) = \int_{\Gamma} y^n d\mu_x(y).$$

Since $\|\mu_x\| \leq M$ for $|x| = 1$, (36) implies that $|P_n(1/x; \alpha)| \leq M$ for $|x| = 1$ and $n = 0, 1, \dots$. Equivalently $|P_n(z; \alpha)| \leq M$ for $|z| = 1$ and hence $\|P_n(z; \alpha)\|_{H^\infty} \leq M$. Since this holds for every $M > \|f\|_{\mathcal{M}_\alpha}$, this proves the theorem. $\square$

The next results generalize the statement made previously concerning the Cesàro sums $\sigma_n(z)$ for a function holomorphic in Δ . Note that $\sigma_n(z) = P_n(z; 2)$ since the binomial coefficient $A_n(2) = n+1$ for $n = 0, 1, \dots$.

Theorem 4.2. Suppose that f is holomorphic in Δ and that $|P_n(z; \alpha)| \leq M$ for $|z| \leq 1$ and $n = 0, 1, \dots$. Then $f \in H^\infty$ and $\|f\|_{H^\infty} \leq M$.

Proof. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $|z| < 1$. Assume that $0 \leq r < 1$ and $|x| = 1$. Then

$$\begin{aligned} \frac{1}{(1-r)^\alpha} f(rx) &= \left\{ \sum_{n=0}^{\infty} A_n(\alpha) r^n \right\} \left\{ \sum_{n=0}^{\infty} a_n r^n x^n \right\} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n A_{n-k}(\alpha) a_k x^k \right) r^n \\ &= \sum_{n=0}^{\infty} A_n(\alpha) P_n(x; \alpha) r^n. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{(1-r)^\alpha} |f(rx)| &\leq \sum_{n=0}^{\infty} A_n(\alpha) |P_n(x; \alpha)| r^n \\ &\leq M \sum_{n=0}^{\infty} A_n(\alpha) r^n = M \frac{1}{(1-r)^\alpha}, \end{aligned}$$

and so $|f(rx)| \leq M$. Since this holds for all r and x , it follows that $|f(z)| \leq M$ for $|z| < 1$. $\square$

The following lemma will be used to establish a partial converse to Theorem 4.2. The kernels $T_n(\theta; \alpha)$ introduced in the lemma are well known, and are studied in [18].

Lemma 4.3. Let $\mu_0 = \frac{1}{2}$ and for $k = 1, 2, \dots$ let $\mu_k(\theta) = \cos k\theta$. Also let

$$T_n(\theta; \alpha) = \frac{1}{A_n(\alpha)} \sum_{k=0}^n A_{n-k}(\alpha) \mu_k(\theta).$$

- (a) If $\alpha \geq 2$ then $T_n(\theta; \alpha) \geq 0$ for $0 \leq \theta \leq 2\pi$ and $n = 0, 1, \dots$.
- (b) If $1 < \alpha < 2$ there is a constant $B(\alpha)$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} |T_n(\theta; \alpha)| d\theta \leq B(\alpha) \quad \text{for } n = 0, 1, \dots$$

Proof. First consider the case $\alpha = 2$. Then (a) is a known fact and the argument for it is as follows. Since $A_n(2) = n+1$ for $n = 0, 1, \dots$,

$$\begin{aligned} T_n(\theta; 2) &= \frac{1}{n+1} \left\{ \frac{n+1}{2} + \sum_{k=1}^n (n-k+1) \cos k\theta \right\} \\ &= \frac{1}{2} \sum_{k=-n}^n \left(1 - \frac{|k|}{n+1} \right) e^{ikt} = \frac{1}{2} \frac{1}{n+1} \left\{ \frac{\sin \frac{n+1}{2} \theta}{\sin \frac{1}{2} \theta} \right\}^2 \geq 0. \end{aligned}$$

This proves (a) when $\alpha = 2$.

Suppose that $\alpha > 0$ and $\beta > 0$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} A_n(\alpha + \beta) z^n &= \frac{1}{(1-z)^{\alpha+\beta}} = \frac{1}{(1-z)^{\alpha}} \frac{1}{(1-z)^{\beta}} \\ &= \sum_{n=0}^{\infty} A_n(\alpha) z^n \sum_{n=0}^{\infty} A_n(\beta) z^n \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^n A_{n-k}(\alpha) A_k(\beta) \right\} z^n. \end{aligned}$$

This shows that

$$(37) \quad A_n(\alpha + \beta) = \sum_{k=0}^n A_{n-k}(\alpha) A_k(\beta).$$

Now assume that $\alpha > 2$. From (37), it follows that

$$\begin{aligned} A_n(\alpha) T_n(\theta; \alpha) &= \sum_{k=0}^n A_{n-k}(\alpha) \mu_k(\theta) \\ &= \sum_{k=0}^n \left\{ \sum_{j=0}^{n-k} A_{n-k-j}(2) A_j(\alpha-2) \right\} \mu_k(\theta) \\ &= \sum_{j=0}^n \left\{ \sum_{k=0}^{n-j} A_{n-j-k}(2) \mu_k(\theta) \right\} A_j(\alpha-2) \\ &= \sum_{j=0}^n T_{n-j}(\theta; 2) A_{n-j}(2) A_j(\alpha-2). \end{aligned}$$

Because $A_{n-j}(2) > 0$, $A_j(\alpha-2) > 0$, and $T_{n-j}(\theta; 2) \geq 0$, this implies that $A_n(\alpha) T_n(\theta; \alpha) \geq 0$. This proves (a) for $\alpha > 2$.

A proof of (b) is contained in [18, Vol. 1, p. 94], where it is shown that the kernel

$$K_n^{\beta}(\theta) = \frac{1}{A_n(\beta+1)} \sum_{k=0}^n A_{n-k}(\beta) D_k(\theta)$$

is “quasipositive” for $0 < \beta < 1$. Here $D_k(\theta)$ denotes the Dirichlet kernel $\frac{1}{2} \sum_{j=-k}^k e^{ij\theta}$. Note that $K_n^{\alpha-1}(\theta) = T_n(\theta; \alpha)$, and since $1 < \alpha < 2$ by assumption, this establishes (b). $\square$

The authors would like to thank B. Muckenhoupt, who provided the proof of (a) for $\alpha > 2$, and who pointed out that this fact is known.

Theorem 4.4. For each $\alpha > 1$ there is a constant $C(\alpha)$ such that if $f \in H^{\infty}$, then

$$(38) \quad \|P_n(z; \alpha)\|_{H^{\infty}} \leq C(\alpha) \|f\|_{H^{\infty}},$$

for $n = 0, 1, \dots$. When $\alpha \geq 2$, (38) holds with $C(\alpha) = 1$.

Proof. The orthonormal relations for the trigonometric functions imply that

$$(39) \quad \frac{1}{2\pi} \int_0^{2\pi} f(ze^{i\theta}) T_n(\theta; \alpha) d\theta = \frac{1}{2} P_n(z; \alpha)$$

for $|z| < 1$.

Suppose that $\alpha \geq 2$, $|z| < 1$, and $f \in H^\infty$. Then (39) and (a) in Lemma 4.3 imply that

$$\frac{1}{2}|P_n(z; \alpha)| \leq \frac{1}{2\pi} \int_0^{2\pi} \|f\|_{H^\infty} T_n(\theta; \alpha) d\theta = \frac{1}{2}\|f\|_{H^\infty}.$$

This proves the theorem in the case $\alpha \geq 2$.

Now suppose that $1 < \alpha < 2$, $|z| < 1$, and $f \in H^\infty$. Then (39) and (b) in Lemma 4.3 imply that

$$\frac{1}{2}|P_n(z; \alpha)| \leq \|f\|_{H^\infty} \frac{1}{2\pi} \int_0^{2\pi} |T_n(\theta; \alpha)| d\theta \leq B(\alpha)\|f\|_{H^\infty}.$$

This proves the theorem where $C(\alpha) = 2B(\alpha)$. $\square$

The assertion in Theorem 4.4 does not hold for $\alpha = 1$. This is because there are functions bounded and holomorphic in Δ such that the sequence of partial sums s_n is not uniformly bounded in Δ [3, p. 444]. Also note that $P_n(z; 1) = s_n(z)$.

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